Graph algorithm notes

Prof Bill - Apr 2020

These notes introduce some cool graph algorithms.

Sections are:

A. Graph search/traversal: DFS, BFS

B. Topological Sort

C. Shortest Path: Dijkstra's Algorithm

D. Minimum Spanning Tree: Prim, Kruskal

QOTD

> Here they come, spinning out of the turn!
> - Phill Georgeff, en.wikipedia.org/wiki/Phil_Georgeff

We are in the home stretch.

thanks… yow, bill

A. Graph Traversal

Reading:

- Sedgewick Ch 4.1 Undirected graphs, algs4.cs.princeton.edu/41graph
- Animation:
  - DFS, www.cs.usfca.edu/~galles/visualization/DFS.html
  - BFS, www.cs.usfca.edu/~galles/visualization/BFS.html

Traversal/search = visit all verts in the graph or all connected verts in a subgraph

Two flavors: Depth-first search (DFS) and Breadth-first search (BFS)

Depth-first search (DFS)

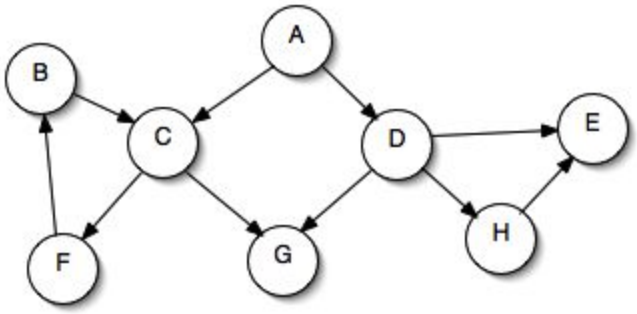
Key concept - it's recursive!

Let's roll the pseudocode:

```
mark all verts in graph as not visited
call dfs( some vert)

// mark vertex v as visited, then recursively visit all connected verts
dfs( vertex v) {
   mark v as visited
   for each vert w adjacent to v {
      if w not visited
         dfs( w)
   }
}
```

Applications of DFS: detect cycle in graph, find path between verts, determine if connected graph, topological sort, determine if bipartite graph, walk thru maze
Source: www.geeksforgeeks.org/?p=11644

Traversal example: Start at vertex A, list verts as you visit them
/* when you have a choice of >1 verts, use sorted order */



DFS traversal example answer: A, C, F, B, G, D, E, H

Breadth-first search (BFS)

Key concept - use a queue! And more pseudocode:

```
// use queue to do a breadth-first traversal of graph
bfs( vertex v) {
   mark all verts not visited
   q = new queue
   q.enqueue( v)
   mark v as visited
   while ! q.isEmpty() {
      v2 = q.dequeue()
      for each vert w: adjacent to v2 {
         if w not visited
            q.enqueue( w)
            mark w as visited
      }
   }
}
```

Applications of BFS: min spanning tree, shortest path, peer-to-peer networks, social media, search engine crawlers,
Source: www.geeksforgeeks.org/applications-of-breadth-first-traversal

Try that traversal example again, using BFS…

BFS traversal example answer: A, C, D, F, G, E, H, B

Question: Hey Prof Bill, got any more DFS and BFS traversal examples to try?

Answer: I do! Head over to our favorite animation site and scroll down to “Graph Algorithms”. The key: try to answer the problem, then run the animation to check your result. www.cs.usfca.edu/~galles/visualization/Algorithms.html

Question: In earlier DFS pseudocode, can we remove recursion?

Answer: Yes! Use a stack, similar to the use of a queue in BFS, www.mathcs.emory.edu/~cheung/Courses/171/Syllabus/11-Graph/dfs.html

B. Topological Sort

Reading:

- ➔ Sedgewick Section 4.2 Directed graphs, algs4.cs.princeton.edu/42digraph
- ➔ Animation: www.cs.usfca.edu/~galles/visualization/TopoSortDFS.html

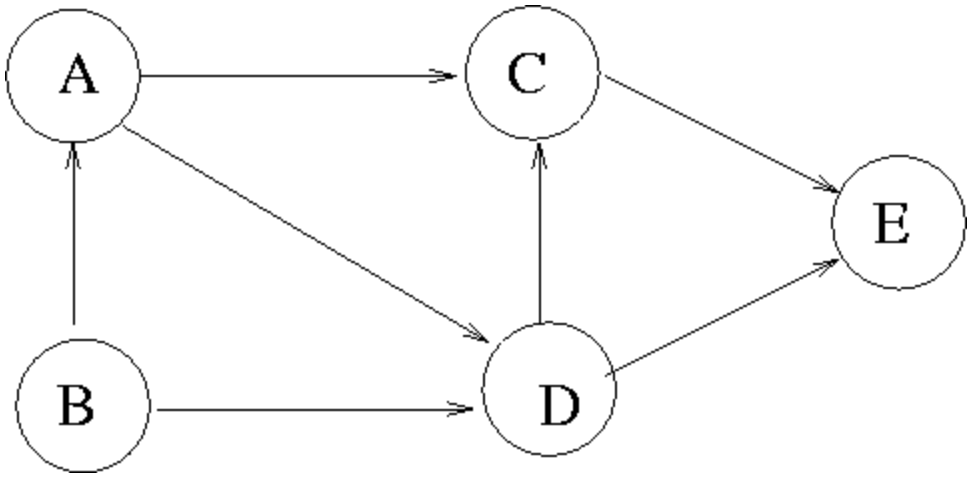
Topological sort - a nice recursive definition

An ordering of vertices in a directed acyclic graph, such that:
If there is a path from u to v, then v appears after u in the ordering.

Also...

- For directed acyclic graph (**DAG**) only, please.
- There can be more than one topological sort for any graph.
- A **cycle** in the graph breaks this definition and therefore has no topological sort.
  For example: path from u to v… and also v to u… then who goes first in sort?

Example: What is topological sort?



Source: lcm.csa.iisc.ernet.in/dsa/node170.html

Example answer: B A D C E

DFS algorithm

Combine DFS with stack. Push vert onto stack after its DFS traversal is done.

Topological sort = reverse postorder of DFS search.

Pseudocode:

```
// print verts in topological sort order
toposort( graph g) {
   s = new stack
   for each vert v in g
      dfs2( v, s)

   while not s.isEmpty()
      v = s.pop()
      print v
}

// dfs traversal, vert pushed on stack once all neighbors visited
dfs2( vertex v, stack s) {
   mark v as visited
   for each vert w: adjacent to v {
      if w not visited
         dfs( w)
   }
   s.push(v)
}
```

C. Shortest Path: Dijkstra's Algorithm

Reading:

- ➔ Sedgewick Section 4.4 Shortest paths, algs4.cs.princeton.edu/44sp
- ➔ Sedgewick slides: algs4.cs.princeton.edu/lectures/keynote/44ShortestPaths-2x2.pdf
- ➔ Animation: www.cs.usfca.edu/~galles/visualization/Dijkstra.html

weighted graph - each edge has a positive weight (distance, force, etc)

Dikstra's Algorithm - single-source shortest path in a graph; greedy + relaxation

- greedy algorithm - choose the shortest (best) edge at each step.
- relaxation - shortest path updated during algorithm with better option, if found

Basis for Dijkstra = "edge relaxation":

```
// if the new path to v is shorter, then use it!
if D[u] + w( u, v) < D[v] then
        D[v] = D[u] + w( u, v)
```

Etc.

- ➢ Positive weights only, negative weights break some algorithms (like Dijkstra)
- ➢ To use Dijkstra on unweighted graphs, use weight=1 for each edge
- ➢ Dijkstra algo has a single source (vert), but find the shortest path to *all* other verts from the source
- ➢ Edsger W. Dijkstra was a GIANT in computer science, en.wikipedia.org/wiki/Edsger_W._Dijkstra

QOTD

> The art of programming is the art of organizing complexity, of mastering multitude and avoiding its bastard chaos as effectively as possible.
> - Dijkstra, en.wikiquote.org/wiki/Edsger_W._Dijkstra

Pseudocode

Prep notes:

- ➔ d[v] = distance (sum of weights) from source to v; init to d[v] = INF
- ➔ prev[v] = previous edge on shortest path to v; init to path[v] = -1
- ➔ d[v] and path[v] are updated and improved over the iterations (relaxation!)
- ➔ controlling data structure: Priority Queue with verts ordered by distance, d[v]

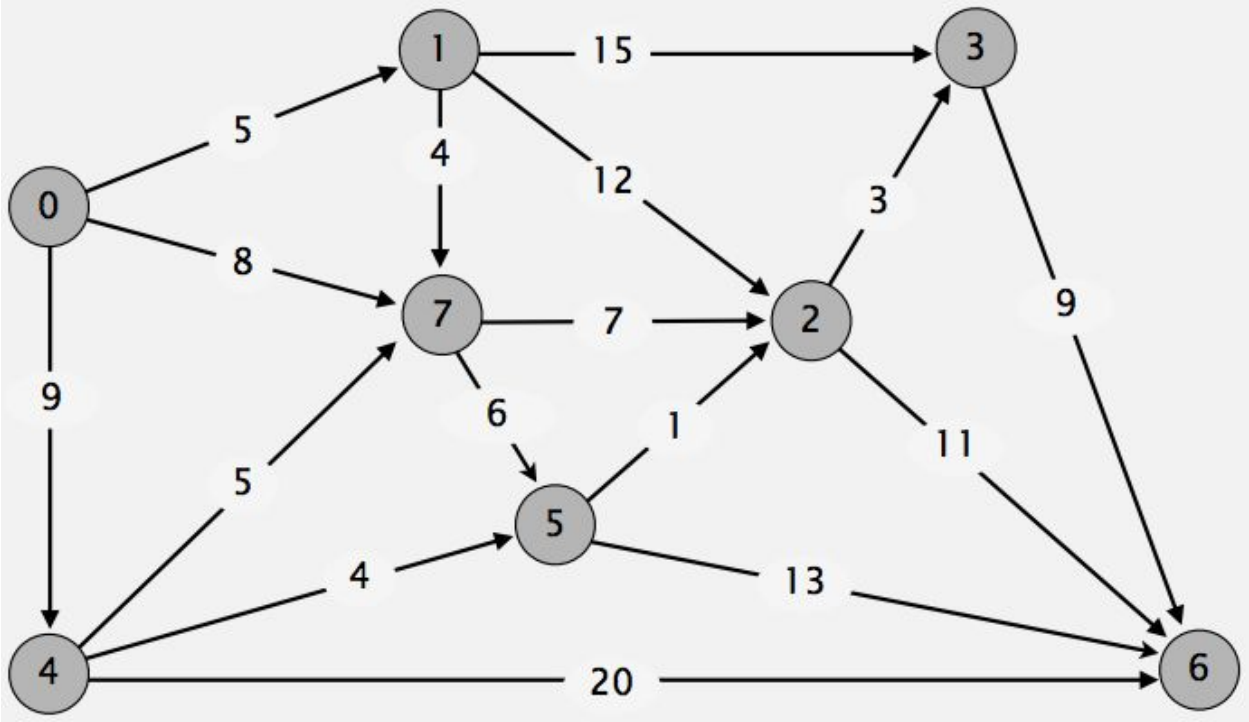
```
// Dijkstra's algorithm, single-source shortest path
shortestPath( graph G, vertex source) {
   for all verts v    // init all vert with infinite distance and no prev edge
      d[v]=INF
      prev[v]=-1

   d[source] = 0    // distance to source is zero
   pq.enqueue(source)

   while pq not empty {
      u = pq.removeMin()
      for each edge (u, v) {
         if d[u] + weight( u, v) < d[v]
            d[v] = d[u] + weight(u, v)    // relaxation
            prev[v] = u
            if pq.contains( v) then pq.decreaseKey( v, d[v])
            else pq.enqueue( v)
      }
   }
}
```

Wrap notes:

- ➔ Shortest path now contained in d[] and prev[] arrays
  - ◆ Shortest path distance from source to any vert v is at d[v]
  - ◆ Reconstruct verts in the path by looping back from prev[v] to the source
- ➔ Performance: With binary heap, O(log V) operations (enqueue, decreaseKey, removeMin) performed for each edge = **O(E log V)**

Question: What are d, prev for an unreachable vertex?

Example (from Sedgewick): Find shortest paths, starting at vert 0



Solution: Sedgewick slides,
algs4.cs.princeton.edu/lectures/keynote/44ShortestPaths-2x2.pdf

Question: Does Dijkstra work if edges have a negative weight?

Dijkstra's algo is greedy. So is Prim's algo for min spanning tree… coming up!

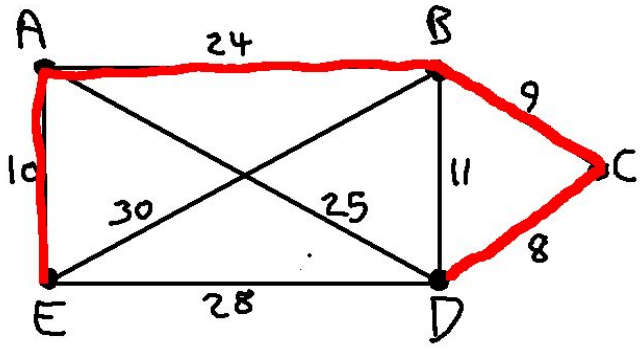
D. Min Spanning Tree: Prim, Kruskal

Reading:

- ❏ Sedgewick Section 4.3 Minimum spanning trees, algs4.cs.princeton.edu/43mst
- ❏ Sedgewick slides, algs4.cs.princeton.edu/lectures/keynote/43MinimumSpanningTrees-2x2.pdf
- ❏ Animation:
  - ❏ Prim's algorithm: www.cs.usfca.edu/~galles/visualization/Prim.html
  - ❏ Kruskal's algorithm: www.cs.usfca.edu/~galles/visualization/Kruskal.html

spanning tree - a tree that contains every vertex in a connected graph (remember - tree means no cycles!); it's a list of edges

min spanning tree - the spanning tree where the sum of edge weights is smallest



Prim's Algorithm

In English: Pick a vertex to be the root of the tree. Find the min weight edge connected to the tree. Add that edge's vertex. Repeat until all vertices are in the tree.

Similar to Dijkstra's Algorithm for finding the shortest path.

Pseudo-code:

```
primsMST( G, startv) {
    create distance array, D[#vertices] = inf
    create parent array, parent[#vertices] = -1
    create known array, known[#vertices] = false

    D[startv] = 0   // start vertex is tree root
    add each D[i] to PriorityQueue PQ
    while PQ not empty {
        u = PQ.removeMin()
        known[u] = true
        for each edge connected to u, (u, v) {
            if ! known[v] and weight of edge < D[v] {
                D[v] = weight of edge
                parent[v] = u
                change D[v] key in PQ   // remove, and re-add to PQ
            }
        }
    }

    for each vertex, v
        add edge ( v, parent[v]) to MST list
}
```

Kruskal's Algorithm

In English:

> sort the edges by weight
> for each edge
> add edge to MST if it doesn't create a cycle

Pseudo-code:

```
kruskalsMST( G) {
    place each vertex in its own disjoint set
    sort all edges in G by weight

    for each edge (u,v) {
        ds1 = find disjoint set of u
        ds2 = find disjoint set of v
        if ds1 != ds2 {
            add edge to MST list
            union( ds1, ds2)   // merge 2 disjoint sets into 1
        }
    }
}
```

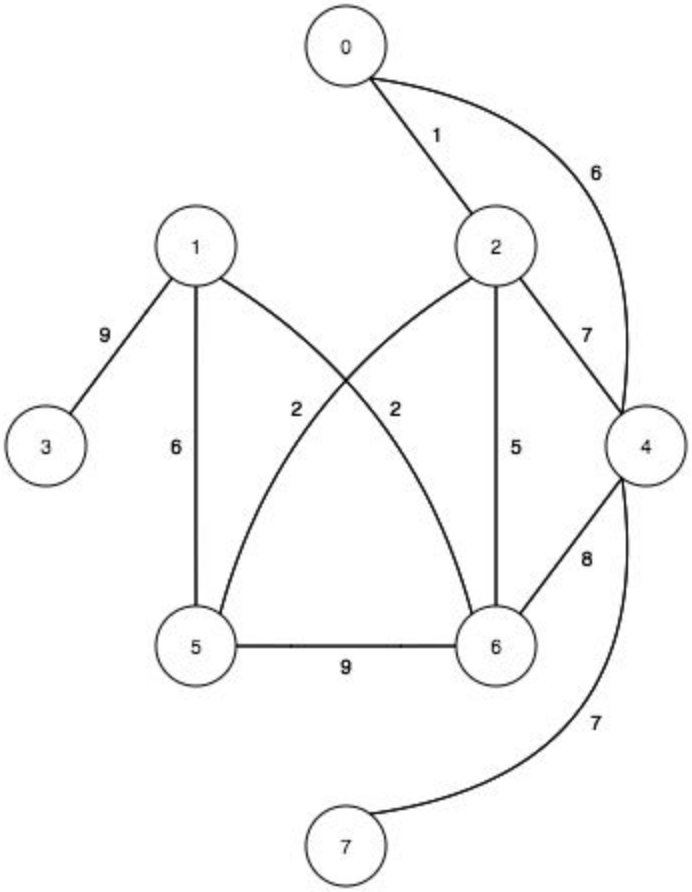
Hey - What are **disjoint sets**? What is **find()**? And **union()**?

Answer: Disjoint set is a collection of sets whose members don't intersect.

We use a nifty representation of disjoint sets (an array) to efficiently determine if adding an edge would create a cycle. A lot of people use disjoint sets...

> en.wikipedia.org/wiki/Disjoint-set_data_structure

Example:



Source: www.cs.usfca.edu/~galles/visualization/Prim.html

1) Run Prim's

2) Run Kruskal's

Then, turn the page.

Answer: